International Journal of Theoretical Physics, Vol. 19, No. 9, 1980

Quantum Mechanics as Relativistic Statistics III. A Relativistic Particle in Two-Dimensional Space-Time

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Received March 13, 1980

Statistics of random world lines in a fixed electromagnetic field are considered. The equation for a vector j^i is obtained. This vector describes the density of random world lines in a pure ensemble. It is shown that in the two-dimensional space-time this equation coincides with the Dirac equation to within the terms of the order of magnitude of $(\lambda/L)^2$ (λ is Compton's wavelength, L is a typical length of the system).

1. INTRODUCTION

In previous papers (Rylov, 1971, 1973, 1977) it has been shown that nonrelativistical quantum mechanics can be considered as a nonrelativistic approximation of world line statistics (relativistic statistics).[1]

There are two notions of the state of the dynamical system. (1) A notion of the nonrelativistical state (n state) is used when the state of a system is given at a certain moment of time. The n state obeys a motion equation, which describes evolution of the n state. For instance, the particle n state is determined at a certain moment by coordinates q and momenta p. (2) A relativistical state (r state) is given over all space-time. The particle r state is the equation of the world line $x^i = q^i(\tau)$. For the deterministic particle the coordinates of its world line obey some equations which are restrictions imposed upon possible r states (see Rylov, 1973).

[1]A comprehensive bibliography on quantum mechanics interpretation from the standpoint of classical mechanics can be found in the survey by Kaliski (1970) and in the book by Belinfante (1973).

The statistics of n states, i.e., statistics of points in the phase space, is the conventional classical statistics. We shall call the statistics of r states, i.e., statistics of world lines (as well as statistics of n dimensional surfaces in the case when the system consists of n particles) as relativistic statistics. The relativistic statistics are statistics of extended directed subjects (lines, surfaces). At this point the relativistic statistics differ from the classical statistics, which are statistics of zero-dimensional subjects (points). A set of peculiarities of relativistic statistics is connected with this circumstance.

The application of relativistic statistics to random (nondeterministic) world lines is developed in papers (Rylov, 1971, 1973, 1977). Here I formulate briefly the main idea. Let there be a classical particle, i.e., a particle whose motion can be described by a world line in the space-time. Let the particle motion be random. For instance, the random (nondeterministic) character of its motion may be conditioned by accidental interaction with a medium (ether). For description of the motion of such a particle it is necessary to use statistical methods, in particular, relativistic statistics.

Application of the statistics, both conventional statistics and relativistic ones, to nondeterministic dynamical systems is determined by a statistical principle (Rylov, 1973).

The Statistical Principle. Any dynamical system S (deterministic or nondeterministic) whose state is described by quantities X corresponds to a deterministic dynamical system A, which is called a statistical ensemble of systems S. The statistical ensemble A is a set of systems S and has the following properties.

(1) A state j of the statistical ensemble A is a state density of systems S.

(2) The equations for the ensemble state j are invariant with respect to transformation $j \to Cj$ ($C = \text{const}$).

(3) If the ensemble state j has proper normalization (on one system) every additive quantity B,[2] attributed to the statistical ensemble as a dynamical system, is the mean value of quantity B for the system S.

(4) If the state density j can be treated as the probability density to detect the system S at the state X, then the mean value $\langle F \rangle$ of any function F of the state X of the dynamical system S can be calculated by means of formula

$$\langle F \rangle = \int F(X) j(X)\, dX$$

where integration is produced over all states X of the system S.

[2]The quantity B is an additive one by definition, if the value of B for several independent dynamical systems is equal to sum of values of B for every system. Examples of additive quantities are: energy, momentum, angular momentum.

The dynamical systems S forming the ensemble A are called elements of the ensemble A.

A formal difference between classical statistics and relativistic ones consists in following. In the classical statistics the state j of the ensemble is a scalar, and point (4) of the statistical principle is always fulfilled. In the relativistic statistics j can be a vector or a tensor. For this reason point (4) of the statistical principle is fulfilled sometimes, and mean values can be calculated for additive quantities only.

2. THE STATEMENT OF THE PROBLEM

Let us consider a classical particle with a mass m and a charge e. The particle moves in the given electromagnetic field with the 4-potential A_i $(i=0,1,2,3)$. The world line $x^i=q^i(\tau)$ $(i=0,1,2,3)$ (τ is a parameter along a world line) is an extremal of the functional of action

$$S[q]=\int_{\min(\tau',\tau'')}^{\max(\tau',\tau'')}\left(-mc\left(\dot{q}^i g_{ik}\dot{q}^k\right)^{1/2}-\frac{\varepsilon e}{c}A_i\dot{q}^i\right)d\tau$$

$$\dot{q}^i\equiv\frac{dq^i}{d\tau} \tag{2.1}$$

Here τ',τ'' are values of τ at the ends of the integration range. The term raised to the $\frac{1}{2}$ power in (2.1) is supposed to be positive. g_{ik} is a metric tensor, which in the inertial frame has a form

$$g_{ik}=\left\|\begin{matrix} c^2 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{matrix}\right\| \tag{2.2}$$

c is the speed of light, ε is a component of the world line orientation.

The world line is supposed to be oriented, i.e., there is some determined direction of motion along the world line. This direction can be described by means of the nonzero vector l^i, which is tangent to the world line and changes continuously along it. On the world line let there be some parametrization P, which is performed by parameter τ (i.e., all points of the world line are numbered by parameter τ). Then

$$l^i=C\frac{dq^i}{d\tau} \tag{2.3}$$

where C is a factor of proportionality. Let us call component ε of the world line orientation $\boldsymbol{\varepsilon}$ with respect to parametrization P as the quantity

$$\varepsilon = \operatorname{sgn} C \tag{2.4}$$

ε takes the values ± 1 and is transformed with the transformation of the parametrization P

$$\tau \to \tau' = \varphi(\tau) \tag{2.5}$$

according to the law

$$\varepsilon \to \varepsilon' = \varepsilon \operatorname{sgn} \frac{\partial \tau'}{\partial \tau} \tag{2.6}$$

If ε is transformed according to (2.6), the integral (2.1) is invariant with respect to any transformation (2.5) of the world line parametrization. Change of the sign of the component ε of the world line orientation (with constant parametrization P) leads to a change of the particle by antiparticle and vice versa.

Usually the world-line orientation is connected with the direction of increase of the parameter τ along the world line. But such a method forbids the transformation (2.5) with $\partial\varphi/\partial\tau < 0$. The method which is used here is more convenient in that respect, that it removes all restrictions upon the manner of parametrization and allows any transformation of the form (2.5).

Orientation $\boldsymbol{\varepsilon}$ is like a usual vector. Components A_i of the vector $\mathbf{A}$ change with transformation from one coordinate system to another, although the vector $\mathbf{A}$ does not change. Likewise ε is a component of an orientation $\boldsymbol{\varepsilon}$ with respect to some parametrization of the world line. With a parametrization transformation the component ε changes, generally speaking, whereas the orientation $\boldsymbol{\varepsilon}$ is invariable.

The electrical charge of the particle, which is described by the action (2.1), is equal to εe, but not e. The electrical charge changes sign with changing the world line orientation (for example, $\varepsilon \to -\varepsilon$). This means that change of the world line orientation transforms a particle into an antiparticle (Rylov, 1970).

Let us consider the statistics of dynamical systems. Each system consists of a particle, which moves in a given electromagnetic field. In accordance with the statistical principle a statistical ensemble corresponds to the dynamical system (2.1). Let us suppose that the world lines of particles have a special property. One and only one world line passes through each point of a region Ω of the space-time. This means that the world lines of the ensemble fill the region Ω without intersection. Such an ensemble is called a simple one in region Ω. The world lines described by the action (2.1) are smooth. For this reason an arbitrary ensemble of such lines can be considered to consist of simple ensembles, i.e., elements of the arbitrary ensemble A are deterministic dynamical systems—the simple ensembles, which in turn consist of uncrossing world lines. Thus consideration of the arbitrary ensemble can be reduced to consideration of the simple ensembles.

Let us consider the properties of the simple ensemble. Such an ensemble can be considered a certain continuous medium. Let us introduce Lagrange's coordinates $\boldsymbol{\xi}=\{\xi_1, \xi_2, \xi_3\}$ numbering the particles of the ensemble. Let us introduce the designation $\xi_0=\tau$. The action of many noninteracting particles is equal to the sum of actions of these particles. For this reason the action of the particle ensemble can be written in the form

$$S[q]=\int_\Omega\left[-mc\left(\frac{\partial q^i}{\partial \xi_0} g_{ik} \frac{\partial q^k}{\partial \xi_0}\right)^{1/2}-\frac{\varepsilon e}{c} A_i(q) \frac{\partial q^i}{\partial \xi_0}\right]|d^4\xi| \tag{2.7}$$

where $d^4\xi=d\xi_0\, d\xi_1\, d\xi_2\, d\xi_3$. Here $q^i=q^i(\xi)$, $\xi=\{\xi_0, \xi_1, \xi_2, \xi_3\}$. The world line of the particle with a number $\boldsymbol{\xi}=\{\xi_1, \xi_2, \xi_3\}$ is given by functions

$$x^i=q^i(\xi_0, \boldsymbol{\xi}) \tag{2.8}$$

with fixed value of $\boldsymbol{\xi}$.

One and only one world line passes through each point x of the region Ω of the space-time. For this reason Lagrange's coordinates $\xi=\{\xi_0, \xi_1, \xi_2, \xi_3\}$ are single-valued functions of coordinates x of the point P in the region Ω, and Jacobian

$$J\equiv\frac{\partial(\xi_0, \xi_1, \xi_2, \xi_3)}{\partial(x^0, x^1, x^2, x^3)}\neq 0 \tag{2.9}$$

Let us consider a vector field

$$j^i=j^i(x)=\varepsilon\varepsilon_0 \frac{\partial J}{\partial \xi_{0,i}}, \qquad i=0,1,2,3 \tag{2.10}$$

where ε is the component of the world line orientation with respect to its parametrization P, which is performed by parameter $\xi_0=\tau$, and

$$\varepsilon_0=\operatorname{sgn} J \tag{2.11}$$

$$\xi_{i,k}\equiv\partial\xi_i/\partial x^k, \qquad i,k=0,1,2,3 \tag{2.12}$$

J is considered a function of $\xi_{i,k}$. By means of the identity

$$\frac{\partial J}{\partial \xi_{l,i}} \equiv J \frac{\partial x^i}{\partial \xi_l} \tag{2.13}$$

the (2.10) can be rewritten in the form

$$j^i = \varepsilon\varepsilon_0 \frac{\partial J}{\partial \xi_{0,i}} = \varepsilon |J| \frac{\partial x^i}{\partial \xi_0} = |J| \varepsilon \frac{\partial x^i}{\partial \tau} \tag{2.14}$$

It follows from (2.14) that j^i represents a vector which is tangent to the world line and does not depend on the parametrization P of the world line. The flux of the vector j^i through the three-dimensional area ds_i can be represented in the form

$$dN = \frac{1}{c} j^i \, ds_i \tag{2.15}$$

Taking into account that according to (2.2) in the inertial frame

$$|g| = |\det\| g_{ik} \|| = c^2 \tag{2.16}$$

one obtains

$$\frac{1}{c} ds_i = \frac{|g|^{1/2}}{c} \frac{\partial(J^{-1})}{\partial x^{i,0}} d\boldsymbol{\xi} = \frac{\partial(J^{-1})}{\partial x^{i,0}} d\boldsymbol{\xi} \tag{2.17}$$

where $J^{-1} = 1/J$ is considered to be a function of quantities $x^{i,k} \equiv \partial x^i / \partial \xi_k$. $d\boldsymbol{\xi}$ is a volume element on the space of quantities $\boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3\}$

$$d\boldsymbol{\xi} = d\xi_1 \, d\xi_2 \, d\xi_3 \tag{2.18}$$

The world line flux through ds_i is obtained in the form

$$dN = \frac{1}{c} j^i \, ds_i = \varepsilon |J| \frac{\partial x^i}{\partial \xi_0} \frac{\partial(J^{-1})}{\partial x^{i,0}} d\boldsymbol{\xi} = \varepsilon\varepsilon_0 \, d\boldsymbol{\xi} \tag{2.19}$$

Let us chose numeration $\boldsymbol{\xi}$ of the world lines in such a way that an unity volume $d\boldsymbol{\xi}$ contains one world line. Then $|dN|$ represents the number of world lines crossing the area ds_i. dN has one sign for particles and another for antiparticles. This means that the vector $j^i = \varepsilon\varepsilon_0 \partial J / \partial \xi_{0,i}$ represents 4-vector of the flux density of particles and/or antiparticles depending on the sign of dN.

Let us consider (2.7) as a functional of functions $\xi_i = \xi_i(x) = \xi_i(q)$, which are reverse functions with respect to $q^i(\xi)$. Then (2.7) takes the form

$$S[\xi,p,j] = \int_\Omega \left[-mc\left(j^i g_{ik} j^k\right)^{1/2} - \frac{\varepsilon e}{c} A_i j^i + p_i\left(\varepsilon\varepsilon_0 \frac{\partial J}{\partial \xi_{0,i}} - j^i\right)\right] |d^4x|, \qquad d^4x = dx^0 dx^1 dx^2 dx^3 \quad (2.20)$$

Here integration is produced over the region Ω of the space-time. p_i are Lagrangian multipliers, which introduce designation (2.10). Owing to (2.9) extremals of the action (2.20), considered as a functional of functions ξ, p, j, coincide with extremals of action (2.7), considered as functional of $q^i(\xi)$.

Thus the action (2.20) describes an evolution of the n state of the simple ensemble. The n state of the simple ensemble is described by the quantities j^i, p_i, ξ_i $(i=0,1,2,3)$. In general, the quantities p_i and ξ_i can be excluded. Then the n state of the simple ensemble is described by vector j^i and its time derivatives. In short, the n state of the simple ensemble of world lines is described by the flux density j^i of the world lines.

An arbitrary ensemble can be considered as an ensemble whose elements are simple ensembles. Generally speaking, an n state of the arbitrary ensemble cannot be described by means of current density j^i and its time derivatives only.

Let us introduce a notion of a pure ensemble. The pure ensemble is an ensemble whose n state can be described by means of quantities j^i and their time derivatives. These quantities obey some equations which describe the n-state evolution. The above-mentioned simple ensemble is a pure one. The reverse statement is not correct, generally speaking. Not all pure ensembles are simple. In other words, an ensemble can be pure and be described by j^i, but it does not consist of uncrossing world lines. An ensemble which is not pure is called a mixed ensemble. The mixed ensemble can be considered as an ensemble whose elements are pure ensembles. The notion of a pure ensemble is important when an ensemble of nondeterministic world lines is considered. In this case the ensemble cannot be a simple one, as far as world lines have random breaks. Apparently, an ensemble of random world lines, which do not cross inside some definite region, does not exist. But an ensemble whose n-state is described by j^i does exist (the pure ensemble).

From the formal standpoint the action (2.20) describes a certain continuous medium: a charged relativistic dust, moving in a given electromagnetic field. A thermal motion of the dust particle is absent.

Let us suppose that besides electromagnetic force some random force acts upon every particle. As a result of the action of this force the motion of the particle becomes random. Something like relativistic Brownian motion arises. The action (2.20) is unsuitable for describing ensembles of such nondeterministic particles. It is necessary to make some assumption about a character of the random force. Taking into account this force one obtains some additional terms in the action (2.20). These terms describe the influence of the chaotic motion of particles upon the mean motion of particles of the ensemble. This mean motion is described by the flux density j^i.

We shall not make any assumption about the random force, but shall make some assumption about the additional term in the expression of action (2.20). It is known from the Brownian motion theory that the influence of chaotic motion upon the mean motion manifests itself in the diffusion of particles. The particles travel from regions of high concentration into regions of low concentration of particles. This phenomenon can be taken into account by adding into (2.20) a term containing derivatives $\partial j^i/\partial x^k$.

Naturally, this term has to contain a combination of derivatives that is relativistically invariant. Besides I want that description to include a possibility of particle–antiparticle pair generation. For this the additional term has to imitate a field that generates pairs. Such a field can be introduced even in classical relativistic mechanics (Rylov, 1970).

For this it is necessary to consider a particle of mass m. Its world line $x^i = q^i(\tau)$ $(i = 0, 1, 2, 3)$ is described by the action

$$S[q] = -\int_{\min(\tau',\tau'')}^{\max(\tau',\tau'')} mc\left[\dot{q}^i g_{ik}\dot{q}^k - \alpha f(q)\right]^{1/2} d\tau \tag{2.21}$$

with $\alpha \to 0$. Here $f(q)$ is an external field, a given function of coordinates. For certain functions $f(q)$ the world line [extremal of functional (2.21)] can turn backward with respect to time. It can describe generation and annihilation of particle–antiparticle pairs. $f(q)$ is a field that generates pairs of particles. A world line must not depend on the manner of numeration of its points. For this reason the action should be invariant with respect to transformation (2.5). This can be achieved tending α to $+0$ in (2.21). Then, generally speaking, limiting $(\alpha \to +0)$ extremals will not be extremals of the limiting $(\alpha \to +0)$ functional, i.e., the generation phenomenon is preserved with $\alpha \to +0$. This can be seen from the fact that the limiting Jacobi–Hamilton equation corresponding to (2.21) has the form

$$f(q)\left(\frac{\partial S}{\partial q^i} g^{ik} \frac{\partial S}{\partial q^k} - m^2c^2\right) = 0 \tag{2.22}$$

At the points where $f(q)=0$, the conventional Jacobi–Hamilton equation

$$\frac{\partial S}{\partial q^i} g^{ik} \frac{\partial S}{\partial q^k} - m^2c^2 = 0 \tag{2.23}$$

can be violated. At these points a break of world line (and, in particular, pair generation or pair annihilation) is possible.

It is worth bearing in mind that the pair generation is connected with introducing the term $\alpha f(q)$ inside the term raised to the $1/2$ power in (2.21). If we expand (2.21) into a power series in α and confine ourselves to the terms of first order, then the pair generation disappears.

Thus, introduction of a field $f(q)$ describing particle generation in the classical approximation is possible. Only a source of this field and its sense are not clear.

Let us add a term into (2.20). For this term to imitate the field $f(q)$, it should be added to the term inside the term raised to the $1/2$ power in (2.20). Let us postulate an action of the pure ensemble in the form

$$S[\xi, p, j] = \int_\Omega \left[-\left(m^2c^2 j^i g_{ik} j^k - \frac{\hbar^2}{8} B_{ik} B^{ik} \right)^{1/2} \right.$$
$$\left. - \frac{\varepsilon e}{c} A_i j^i + p_i \left(\varepsilon \varepsilon_0 \frac{\partial J}{\partial \xi_{0,i}} - j^i \right) \right] |d^4x| \tag{2.24}$$

where

$$B_{ik} = j_{i,k} - j_{k,i}, \qquad B^{ik} = g^{il} g^{ks} B_{ls} \tag{2.25}$$

$\hbar$ is Planck's constant. The comma denotes differentiation:

$$j_{i,k} \equiv \partial j_i / \partial x^k$$

The additional term is universal in the sense that it contains universal constants c, $\hbar$, and derivatives of j^i only. It does not contain any parameters that describe the particle. The additional term is constructed from flux density components j^i in the same way the Lagrangian of the electromagnetic field is constructed from components of the 4-potential A_i. The expression in square brackets in (2.24) is generally covariant and has the same form in any curvilinear coordinate system.

3. EQUATIONS OF MOTION

Let us represent (2.24) in the form

$$S[\xi,p,j] = \int_\Omega \left[-mcK - \frac{\varepsilon e}{c} A_i j^i + p_i\left(\varepsilon\varepsilon_0 \frac{\partial J}{\partial \xi_{0,i}} - j^i \right)\right] |d^4x| \tag{3.1}$$

where

$$K = \left(j^i g_{ik} j^k - \frac{\lambda^2}{8} B_{ik} B^{ik} \right)^{1/2} \tag{3.2}$$

and

$$\lambda = \frac{\hbar}{mc} \tag{3.3}$$

is a Compton's wavelength of the particle. Varying the action (3.1) with respect to ξ_i leads to the following equations of motion:

$$\frac{\delta S}{\delta \xi_i} = -\partial_k \left(p_l \varepsilon\varepsilon_0 \frac{\partial^2 J}{\partial \xi_{0,l} \partial \xi_{i,k}} \right) = 0 \tag{3.4}$$

where

$$\partial_k \equiv \partial / \partial x^k$$

Varying with respect to p_i leads to equations (2.10). It is convenient to exclude Lagrangian variables ξ_i and to write equations in terms of Eulerian variables. Let us use the following identities:

$$\frac{\partial}{\partial x^i} \frac{\partial J}{\partial \xi_{k,i}} \equiv 0, \qquad \frac{\partial}{\partial x^k} \frac{\partial^2 J}{\partial \xi_{s,l} \partial \xi_{i,k}} \equiv 0 \tag{3.5}$$

$$\frac{\partial^2 J}{\partial \xi_{0,l} \partial \xi_{i,k}} \xi_{i,s} \equiv \frac{\partial J}{\partial \xi_{0,l}} \delta_s^k - \frac{\partial J}{\partial \xi_{0,k}} \delta_s^l \tag{3.6}$$

By means of (2.10) the first identity of (3.5) can be written in the form

$$\partial_i \left(\frac{1}{\varepsilon\varepsilon_0} j^i \right) = 0 \tag{3.7}$$

Multiplying (3.4) by $\xi_{i,s}$ and using identities (3.5), (3.6) one obtains

$$\frac{1}{\varepsilon\varepsilon_0} j^l \{ \partial_k(\varepsilon\varepsilon_0 p_l) - \partial_l(\varepsilon\varepsilon_0 p_k) \} = 0 \tag{3.8}$$

As far as $\varepsilon\varepsilon_0 = \pm 1$, the equations (3.7),(3.8) can be rewritten in the form

$$\partial_i j^i = 0 \tag{3.9}$$

$$j^l(\partial_k p_l - \partial_l p_k) = 0 \tag{3.10}$$

The equations (3.9),(3.10) are fulfilled everywhere except, perhaps be, those points where $\varepsilon\varepsilon_0$ changes sign.

Varying with respect to j^i leads to the equations

$$p_i = -\frac{mc}{K} j_i - \frac{\lambda^2 mc}{4} \partial_k \left(\frac{g^{kl} B_{il}}{K} \right) - \frac{\varepsilon e}{c} A_i \qquad i = 0,1,2,3 \tag{3.11}$$

Finally, let us write those equations (2.25) that contain a time derivative ∂_0:

$$\partial_0 j_\alpha = -B_{0\alpha} + \partial_\alpha j_0, \qquad \alpha = 1,2,3 \tag{3.12}$$

The system of equations (3.9)–(3.12) contains 10 independent equations of the first order. It contains 10 quantities: j^i $(i=0,1,2,3)$, p_α, $B_{0\alpha}$ $(\alpha = 1,2,3)$. As far as quantities p_0 and $B_{\alpha\beta}$ $(\alpha, \beta = 1,2,3)$ are concerned, these quantities can be calculated through j^i $(i=0,1,2,3)$, p_α, $B_{0\alpha}$ $(\alpha = 1,2,3)$ given at a definite time moment. The p_0 is expressed by means of (3.11) with $i=0$. The $B_{\alpha\beta}$ are expressed by means of (2.25) with $i, k = 1,2,3$. These expressions contain only space derivatives. Thus, the n state of the ensemble is determined by j^i $(i=0,1,2,3)$, p_α, $B_{0\alpha}$ $(\alpha = 1,2,3)$. Besides this it is necessary to fix the sign of the quantity K defined by (3.2), because K is defined by (3.2) to within the sign.

In the case when $\lambda = 0$ and the action (3.1) is reduced to the form (2.20), the problem of the sign of K is solved easily. K should be chosen positive, because the action has a minimum in this case only. Besides that, the sign of K is conserved along lines of the vector j^i (along world lines) owing to the equation of motion.

In the case $\lambda \neq 0$ the problem of choice of the sign of K becomes complicated. Although the condition $K \geqslant 0$ can be imposed and this corresponds to the minimum of the action (3.1), whether the sign of K is conserved because of the equations of motion remains open.

Let $\phi(x) = \text{const}$ be the equation of the characteristic surface of the system of equations (3.9)–(3.12). Then the characteristic equation of the system (3.9)–(3.12) can be represented in the form

$$\frac{\lambda^6}{K^3} (\phi_i \phi^i)^3 (\phi_k j^k)^2 \left((\phi_l \phi^l) + \frac{\lambda^2}{4K^2} (\phi_s B^s_{\cdot m} B^{rm} \phi_r) \right) = 0 \tag{3.13}$$

where

$$B^{s}_{\cdot m} \equiv g^{sl} B_{lm}, \qquad B^{ik} = B^{i}_{\cdot l} g^{lk}$$

and the vector

$$\phi_i \equiv \partial_i \phi, \qquad \phi^i = g^{ik} \phi_k \tag{3.14}$$

describes a normal of the characteristic surface. The characteristic equation (3.13) is Lorentz invariant. Every null direction ϕ_i (i.e., $\phi_i \phi^i = 0$) is characteristic (threefold degeneracy). Every direction ϕ_i, which is orthogonal to j^i, is characteristic (twofold degeneracy). Finally, the direction ϕ_i, which obeys the equation

$$\phi_i \tilde{g}^{ik} \phi_k = 0 \tag{3.15}$$

where

$$\tilde{g}^{ik} = g^{ik} + \frac{\lambda^2}{4K^2} B^{i}_{\cdot l} B^{kl} \tag{3.16}$$

is characteristic. The equation (3.15) has real solutions (see Appendix). Thus all characteristics of the system (3.9)–(3.12) are real.

Let us write the expression of the canonical energy–momentum tensor, which is defined by the relation

$$T^{i}_{\cdot k} = \sum_s \frac{\partial L}{\partial u_{s,i}} u_{s,k} - L \delta^i_k \tag{3.17}$$

where L is the Lagrangian density in the action (3.1), and u_s denotes a set of variables which are variated in (3.1): $u = \{\xi, p, j\}$. Doing the calculation and omitting terms that have the form of a divergence, one obtains

$$T^{i}_{\cdot k} = mc\left(K\delta^i_k + \frac{1}{K}(j^i j_k - j^l j_l \delta^i_k) - \frac{\lambda^2}{4K}(B_{kl} j^{i,l} + B^{li} j_{l,k})\right) + \frac{\varepsilon e}{c} A_k j^i \tag{3.18}$$

In particular, the energy density is expressed in the form

$$T^{0}_{\cdot 0} = mc\left(K + \frac{j_\alpha j_\alpha}{K} + \frac{\lambda^2}{4c^2 K} B_{0\alpha} B_{0\alpha}\right) + \frac{\varepsilon e}{c} A_0 j^0 \tag{3.19}$$

It follows from (3.19) that in the absence of an electromagnetic field the sign of the energy density coincides with the sign of K. This means that we should consider K to be positive:

$$K > 0 \tag{3.20}$$

Apparently, the condition (3.20) should be taken even if it happens that the sign of K is not an integral of the equations of motion (3.9)–(3.12). In this case the (3.20) should be considered as an additional condition in the search of an extremum of the action (3.1).

Later on, investigating properties of the system (3.9)–(3.12), one assumes for generality that K can have any sign.

4. TWO-DIMENSIONAL CASE

Let us consider the two-dimensional case (one time coordinate and one space coordinate). Then the system (3.9)–(3.12) can be written in the form

$$j^0(\partial_1 p_0 - \partial_0 p_1) = 0 \tag{4.1}$$

$$p_1 = -\frac{mc}{K} j_1 + \frac{\lambda^2 mc}{4c^2} \partial_0\left(\frac{B_{01}}{K}\right) - \frac{\varepsilon e}{c} A_1 \tag{4.2}$$

$$p_0 = -\frac{mc}{K} j_0 + \frac{\lambda^2 mc}{4} \partial_1\left(\frac{B_{01}}{K}\right) - \frac{\varepsilon e}{c} A_0 \tag{4.3}$$

$$\frac{1}{c^2} \partial_0 j_0 - \partial_1 j_1 = 0 \tag{4.4}$$

$$\partial_0 j_1 = -B_{01} + \partial_1 j_0 \tag{4.5}$$

The solution of the equation (4.1) can be represented in the form

$$p_0 = \partial_0 S, \qquad p_1 = \partial_1 S \tag{4.6}$$

where S is a certain function of coordinates $x = \{x^0, x^1\}$ having the dimensionality of an action.

Substituting (4.6) into (4.2) and into (4.3) one obtains

$$\partial_1 S = -\frac{mc}{K} j_1 + \frac{\lambda^2 m}{4c} \partial_0\left(\frac{B_{01}}{K}\right) - \frac{\varepsilon e}{c} A_1 \tag{4.7}$$

$$\partial_0 S = -\frac{mc}{K} j_0 + \frac{\lambda^2 mc}{4} \partial_1\left(\frac{B_{01}}{K}\right) - \frac{\varepsilon e}{c} A_0 \tag{4.8}$$

Owing to (2.2) one obtains

$$K=\left(\frac{1}{c^2}j_0^2-j_1^2+\frac{\lambda^2}{4c^2}B_{01}^2\right)^{1/2} \tag{4.9}$$

Thus the system of equations (4.4), (4.5), (4.7), (4.8) of the first order is obtained. It contains variables j_0, j_1, S, B_{01}.

The characteristic equation of the system (4.4), (4.5), (4.7), (4.8) has a form

$$\lambda^2 m\left(\frac{1}{c^2}\Phi_0^2-\Phi_1^2\right)^2=0, \qquad \Phi_i\equiv\frac{\partial\Phi}{\partial x^i} \tag{4.10}$$

A solution of the (4.10) can be represented as

$$x\pm ct=\text{const}, \qquad x\equiv x^1, \qquad t\equiv x^0 \tag{4.11}$$

with every characteristic being bicharacteristic. Apparently a direct integration of the system (4.4)–(4.8) is impossible. Essentially those solutions are of interest that describe a bunch with a size $L\gg\lambda$. The flux components $\{j_0, j_1\}$ are supposed to vary slowly inside the bunch, so that

$$\lambda|\partial_1 j_0|\approx\left|\frac{\lambda j_0}{L}\right|\ll j_0$$

$$\lambda|\partial_0 j_1|\approx\frac{\lambda c}{L}|j_1|\ll c|j_1| \tag{4.12}$$

If besides this $c^{-2}j_0^2\simeq j_1^2\simeq c^{-2}j_0^2-j_1^2$, then together with (4.9) these estimations give

$$\frac{\lambda^2B_{01}^2}{c^2}\ll|j_1|^2, \qquad \frac{\lambda^2B_{01}^2}{c^2}\ll\frac{1}{c^2}j_0^2, \qquad \left|\frac{\lambda B_{01}}{c}\right|\ll K \tag{4.13}$$

This allows one to neglect the terms that contain λ in equations (4.7)–(4.9). The equations (4.7)–(4.8) can be written in the zeroth approximation in the form

$$\partial_1 S=-\frac{mc}{K_0}j_1-\frac{\varepsilon e}{c}A_1 \tag{4.14}$$

$$\partial_0 S=-\frac{mc}{K_0}j_0-\frac{\varepsilon e}{c}A_0 \tag{4.15}$$

$$K_0=\left(\frac{1}{c^2}j_0^2-j_1^2\right)^{1/2} \tag{4.16}$$

The system of equations (4.4),(4.14)–(4.16) describes a motion of an ensemble of classical particles in an external electromagnetic field. This description is satisfactory until inequalities (4.13) are fulfilled.

Let the ensemble be a bunch of particles with the same momentum p and energy E. Let the size of the bunch be $L \gg \lambda$. The system of equations (4.4)–(4.16) describes a motion of the bunch satisfactory everywhere except the small regions near the turning point. The fact is, that in the strong enough electromagnetic field there can be regions which cannot be achieved by the particles with energy E. The boundary of such a region (here it is one dimensional) is the turning point. Reaching this point, the bunch of particles reflects and travels backward. At one side of the turning point $j_0 = 0$, and at the other side $j_0 \neq 0$. Thus near the turning point the gradient of j_0 is large, and conditions (4.13) are not fulfilled. It is necessary to take into account the neglected terms.

Let us return to consideration of the system (4.4)–(4.9). Let us introduce dimensionless quantities (phases)

$$\varphi = \frac{S}{\hbar}, \qquad \kappa = \frac{\lambda B_{01}}{2cK} \tag{4.17}$$

Let us consider the case when the phase κ is small. Let us try to find a system of linear equations which differs from the system (4.4)–(4.9) slowly, if κ is small.

Adding and subtracting (4.7) and (4.8), one obtains

$$\lambda\left(\frac{1}{c}\partial_0 + \partial_1\right)\left(\varphi - \frac{\kappa}{2}\right) - \frac{\varepsilon e}{mc^2}\left(\frac{1}{c}A_0 + A_1\right) = -\frac{1}{K}\left(\frac{1}{c}j_0 + j_1\right) \tag{4.18}$$

$$\lambda\left(\frac{1}{c}\partial_0 - \partial_1\right)\left(\varphi + \frac{\kappa}{2}\right) - \frac{\varepsilon e}{mc^2}\left(\frac{1}{c}A_0 - A_1\right) = -\frac{1}{K}\left(\frac{1}{c}j_0 - j_1\right) \tag{4.19}$$

Adding and subtracting (4.4) and (4.5), one obtains

$$\left(\frac{1}{c}\partial_0 + \partial_1\right)\left(\frac{1}{c}j_0 - j_1\right) = \frac{1}{c}B_{01} \tag{4.20}$$

$$\left(\frac{1}{c}\partial_0 - \partial_1\right)\left(\frac{1}{c}j_0 + j_1\right) = -\frac{1}{c}B_{01} \tag{4.21}$$

Let us introduce designations

$$W_+ = \left(\frac{1}{c}j_0 + j_1\right)^{1/2}, \qquad W_- = \left(\frac{1}{c}j_0 - j_1\right)^{1/2} \tag{4.22}$$

$$\partial_+ = \frac{1}{c}\partial_0 + \partial_1, \qquad \partial_- = \frac{1}{c}\partial_0 - \partial_1 \tag{4.23}$$

$$\mu_+ = \frac{S}{\hbar} + \frac{\lambda}{4c}\frac{B_{01}}{K} = \varphi + \frac{\kappa}{2}, \qquad \mu_- = \frac{S}{\hbar} - \frac{\lambda}{4c}\frac{B_{01}}{K} = \varphi - \frac{\kappa}{2} \tag{4.24}$$

If j^i is a timelike vector, i.e., $|c^{-1}j_0|>|j_1|$, then W_+, W_- are real for $j_0>0$ and W_+, W_- are imaginary for $j_0<0$. One obtains from (4.9)

$$K=\frac{W_+W_-}{(1-\kappa^2)^{1/2}} \tag{4.25}$$

In terms of (4.22)–(4.24) the equations (4.18), (4.19) take the form

$$\lambda W_-\partial_+\mu_- - \frac{\varepsilon e}{mc^2}\left(\frac{1}{c}A_0+A_1\right)W_- + W_+(1-\kappa^2)^{1/2}=0 \tag{4.26}$$

$$\lambda W_+\partial_-\mu_+ - \frac{\varepsilon e}{mc^2}\left(\frac{1}{c}A_0-A_1\right)W_+ + W_-(1-\kappa^2)^{1/2}=0 \tag{4.27}$$

The equations (4.20), (4.21) can be represented in the form

$$\lambda\partial_+W_- = \frac{\kappa}{(1-\kappa^2)^{1/2}}W_+ \tag{4.28}$$

$$\lambda\partial_-W_+ = -\frac{\kappa}{(1-\kappa^2)^{1/2}}W_- \tag{4.29}$$

If W_+, W_- are real (imaginary), then owing to (4.24) the equations (4.26) and (4.28) represent to within κ, respectively, real (imaginary) and imaginary (real) parts of the equation

$$-i\lambda\partial_+\left(W_-e^{i\mu_-}\right) - \frac{\varepsilon e}{mc^2}\left(\frac{1}{c}A_0+A_1\right)W_-e^{i\mu_-} + W_+e^{i\mu_+}=0 \tag{4.30}$$

Likewise, if W_+, W_- are real (imaginary), then the (4.27), (4.29) represent to within κ, respectively, real (imaginary) and imaginary (real) parts of the equation

$$-i\lambda\partial_-\left(W_+e^{i\mu_+}\right) - \frac{\varepsilon e}{mc^2}\left(\frac{1}{c}A_0-A_1\right)W_+e^{i\mu_+} + W_-e^{i\mu_-}=0 \tag{4.31}$$

The equations (4.30), (4.31) are linear with respect to variables $W_+\exp(i\mu_+)$, $W_-\exp(i\mu_-)$. This means that in the first approximation with respect to quantity κ the system (4.26)–(4.29) can be reduced to a system of linear equations.

However, using complex variables $W_+ \exp(i\mu_+), W_- \exp(i\mu_-)$ is possible only in case W_+ and W_- are real or imaginary simultaneously. In this case only the equations (4.30), (4.31) are equivalent to the system (4.26)–(4.29) to within κ. In the case when one of quantities W_+ or W_- is real and another one is imaginary, this equivalence is destroyed. In the common case instead of the imaginary unit i its matrix analog τ can be used. τ is a two-dimensional matrix

$$\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tau\tau = -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.32}$$

τ has the same properties as the imaginary unit i has. τ is another imaginary unit. It differs from the imaginary unit i, which can arise in expressions of W_+, W_-. For this reason these imaginary units can be separated.

Let us introduce new variables

$$\varphi_+ = \frac{1}{2^{1/2}} W_+ e^{\tau\mu_+} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} \dfrac{W_+}{2^{1/2}}(\cos\mu_+ + \sin\mu_+) \\ \dfrac{W_+}{2^{1/2}}(\cos\mu_+ - \sin\mu_+) \end{bmatrix}$$

$$\varphi_- = \frac{1}{2^{1/2}} W_- e^{\tau\mu_-} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} \dfrac{W_-}{2^{1/2}}(\cos\mu_- + \sin\mu_-) \\ \dfrac{W_-}{2^{1/2}}(\cos\mu_- - \sin\mu_-) \end{bmatrix} \tag{4.33}$$

In terms of φ_+, φ_- the equations (4.26)–(4.29) can be written in the form

$$\left[-\lambda\tau\partial_+ - \frac{\varepsilon e}{mc^2}\left(\frac{1}{c}A_0 + A_1\right)\right]\varphi_- + e^{-\tau\kappa}\left((1-\kappa^2)^{1/2} + \frac{\kappa}{(1-\kappa^2)^{1/2}}\tau\right)\varphi_+ = 0$$

$$\left[-\lambda\tau\partial_- - \frac{\varepsilon e}{mc^2}\left(\frac{1}{c}A_0 - A_1\right)\right]\varphi_+ + e^{\tau\kappa}\left((1-\kappa^2)^{1/2} - \frac{\kappa}{(1-\kappa^2)^{1/2}}\tau\right)\varphi_- = 0$$

(4.34)

If κ is small the equations represent to within κ^2 linear equations with respect to variables φ_+, φ_-.

Let us introduce now the four-line column ψ

$$\psi=\frac{1}{2^{1/2}}\begin{pmatrix}\varphi_+\\ \varphi_-\end{pmatrix}=\frac{1}{2^{1/2}}\begin{bmatrix}\left(\frac{1}{c}j_0+j_1\right)^{1/2}\cos\left(\frac{S}{\hbar}+\frac{\lambda}{4c}\frac{B_{01}}{K}-\frac{\pi}{4}\right)\\ \left(\frac{1}{c}j_0+j_1\right)^{1/2}\sin\left(\frac{S}{\hbar}+\frac{\lambda}{4c}\frac{B_{01}}{K}-\frac{\pi}{4}\right)\\ \left(\frac{1}{c}j_0-j_1\right)^{1/2}\cos\left(\frac{S}{\hbar}-\frac{\lambda}{4c}\frac{B_{01}}{K}-\frac{\pi}{4}\right)\\ \left(\frac{1}{c}j_0-j_1\right)^{1/2}\sin\left(\frac{S}{\hbar}-\frac{\lambda}{4c}\frac{B_{01}}{K}-\frac{\pi}{4}\right)\end{bmatrix}$$

$$=\frac{1}{2^{1/2}}\begin{bmatrix}W_+\cos\left(\varphi+\frac{\kappa}{2}-\frac{\pi}{4}\right)\\ W_+\sin\left(\varphi+\frac{\kappa}{2}-\frac{\pi}{4}\right)\\ W_-\cos\left(\varphi-\frac{\kappa}{2}-\frac{\pi}{4}\right)\\ W_-\sin\left(\varphi-\frac{\kappa}{2}-\frac{\pi}{4}\right)\end{bmatrix} \tag{4.35}$$

and four-line real matrices

$$\gamma^0=\frac{1}{c}\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},\qquad \gamma^1=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix},\qquad \tau_1=\begin{pmatrix}\tau & 0\\ 0 & \tau\end{pmatrix}$$

$$\nu=c\gamma^1\gamma^0\tau_1=\begin{pmatrix}\tau & 0\\ 0 & -\tau\end{pmatrix},\qquad \tau_0=\begin{bmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix} \tag{4.36}$$

I is a unit matrix. The matrices satisfy the following relations:

$$\gamma^i\gamma^k+\gamma^k\gamma^i=2g^{ik}I,\qquad i,k=0,1$$

$$\gamma^i\tau_k=\tau_k\gamma^i,\qquad \nu\gamma^i=-\gamma^i\nu,\qquad i,k=0,1$$

$$\tau_1\nu=\nu\tau_1,\qquad \tau_0\nu=-\nu\tau_0,\qquad \tau_1^2=-I,\qquad \tau_0^2=I \tag{4.37}$$

Taking into account (3.3), the equations (4.34) can be written in the form

$$\left[\gamma^i\left(-\hbar\tau_1\partial_i-\frac{\varepsilon e}{c}A_i\right)+\alpha mc\right]\psi-\alpha Mc\psi=0 \tag{4.38}$$

Where M is a matrix:

$$M = m\left[1 - |1-\kappa^2|^{-1/2} e^{-\nu\kappa}(1+\nu\kappa-\kappa^2)\right] = m\left(\frac{\nu\kappa^3}{3} + \frac{5}{6}\kappa^4 + O(\kappa^4)\right) \tag{4.39}$$

$$\alpha = \frac{|(1-\kappa^2)^{1/2}|}{(1-\kappa^2)^{1/2}} = \frac{|W_+ W_-|}{W_+ W_-} \operatorname{sgn} K \tag{4.40}$$

α can take four possible values: $\pm 1, \pm i$. Real values of α correspond to $\kappa^2 < 1$ (timelike vector j^i). Imaginary values of α correspond to $\kappa^2 > 1$ (spacelike vector j^i). The sign of α is determined ambiguously by the quantities $j^i, \varphi, B_{01}, \operatorname{sgn} K$. In this sense the $\operatorname{sgn}\alpha$ is an independent variable, which should be given at the initial moment. Such an ambiguous determination of α is connected with the ambiguous determination of the column ψ by means of (4.35). Really, the (4.35) contains two double-valued functions W_+ and W_-, which are determined by (4.22). If $j_0, j_1, \varphi, \kappa$ are fixed, then at least four different ways of determining ψ exist. The four ways can be obtained from one of them by means of a combination of the following two transformations:

$$T_1: \psi \to \psi' = -\psi, \qquad \alpha \to \alpha' = \alpha, \qquad W_+ \to -W_+, \qquad W_- \to -W_- \tag{4.41}$$

$$T_2: \psi \to \psi' = c\gamma^0\gamma^1\psi, \qquad \alpha \to \alpha' = -\alpha, \qquad W_+ \to -W_+, W_- \to W_- \tag{4.42}$$

The variables $j^i, \varphi, \kappa, \operatorname{sgn} K$ are not changed by transformations (4.41), (4.42). The equation (4.38) is invariant with respect to transformation (4.41), (4.42).

Let

$$\psi_1 = \psi_1(j_0, j_1, \varphi, \kappa) \tag{4.43}$$

be a certain way of determining the column ψ. Then

$$\psi_2 = T_1\psi_1, \qquad \psi_3 = T_2\psi_1, \qquad \psi_4 = T_2T_1\psi_1 \tag{4.44}$$

are another three possible ways of determining the column ψ. The following relations hold:

$$(T_1T_2 - T_2T_1)\psi = 0, \qquad T_1^2\psi = \psi, \qquad T_2^2\psi = \psi \tag{4.45}$$

For this reason the transformation of the form $T_1^{s_1}T_2^{s_2}\psi$ (s_1, s_2 are integers) does not lead to new ways of determining ψ.

The reverse transformation from ψ to variables $j_0, j_1, \varphi, \kappa$ can be represented in the form

$$j^0=\bar{\psi}\gamma^0\psi, \qquad j^1=\bar{\psi}\gamma^1\psi \tag{4.46}$$

$$\tan(2\varphi)=\frac{\bar{\psi}\tau_1\tau_0\psi}{\bar{\psi}\tau_0\psi}, \qquad \tan\kappa=-\frac{\bar{\psi}\nu\psi}{\bar{\psi}\psi}$$

$$\operatorname{sgn}K=\alpha\frac{W_+W_-}{|W_+W_-|}=\alpha\left|\frac{\bar{\psi}\psi}{\cos\kappa}\right|\frac{\cos\kappa}{\bar{\psi}\psi} \tag{4.47}$$

where

$$\bar{\psi}=c\tilde{\psi}\gamma^0 \tag{4.48}$$

and $\tilde{\psi}$ represents a line obtained by the transposition of the column ψ.

It is easy to see that the relations (4.46),(4.47) are invariant with respect to transformations (4.41),(4.42). Beyond this the equation (4.38) and relations (4.46)–(4.47) are invariant with respect to transformation:

$$\psi\to\psi'=e^{\tau_1 a}\psi, \qquad \alpha\to\alpha'=\alpha \tag{4.49}$$

where a is an arbitrary real number. Thus, uncertainty of the choice of the column ψ increases. ψ can be chosen to within a constant phase factor $\exp(\tau_1 a)$. (4.41) is obtained by $a=\pi$.

It follows from (4.47) that ψ determines phases φ and κ to within an additive constant only ($s\pi/2$ and $n\pi$, s and n are integers). The ambiguity of φ is unessential, because according to (4.6) and (4.17) the φ is a potential of p_i. The constant additive term gives no contribution into p_i. But the ambiguity of κ is essential. First of all let us note that κ is determined unambiguously by the second relation (4.47) in that case when j^i is a timelike vector, i.e.,

$$\frac{1}{c^2}j_0^2-j_1^2>0 \tag{4.50}$$

In fact, it follows from (4.9) and (4.17) that

$$B_{01}=\frac{2c}{\lambda}\frac{\kappa\left[(1/c^2)j_0^2-j_1^2\right]^{1/2}}{(1-\kappa^2)^{1/2}} \tag{4.51}$$

It follows from this that in the case (4.50) with real B_{01} the κ can change inside the interval $(-1, 1)$ only. This condition together with (4.47) determines κ single valuedly.

If the vector j^i is spacelike and B_{01} is real, then it follows from (4.51) that $|\kappa|>1$. In this case the phase κ is determined from (4.47) ambiguously.

Let us solve (ambiguously) the second relation (4.47) with respect to κ:

$$\kappa = -\arctan \frac{\bar{\psi}\nu\psi}{\bar{\psi}\psi} + n\pi \tag{4.52}$$

where n is an integer variable. Substituting (4.52) into (4.39), one obtains an expression of the matrix M through ψ. This permits one to consider the equation (4.38) as an equation for the determination of the four components of the function ψ. Generally speaking, this equation is nonlinear, because M depends on ψ. But, if $|\kappa| \ll 1$, then according to (4.39) $M \ll m$, and the last term of (4.38) can be neglected. In this case the linear equation

$$\gamma^i\left(-\hbar\tau_1\partial_i - \frac{\varepsilon e}{c}A_i\right)\psi + \alpha mc\psi = 0 \tag{4.53}$$

arises where $\alpha = \pm 1$, because $|\kappa| \ll 1$.

A solution of the equation (4.53) can be obtained by combining solutions of two linear equations

$$\gamma^i\left(-\hbar\tau_1\partial_i - \frac{\varepsilon e}{c}A_i\right)\psi + mc\psi = 0 \tag{4.54}$$

$$\gamma^i\left(-\hbar\tau_1\partial_i - \frac{\varepsilon e}{c}A_i\right)\psi - mc\psi = 0 \tag{4.55}$$

Which of the two equations (4.54), (4.55) should be taken in a given region is determined by the sign of the variable α from (4.40), i.e., it is determined essentially by choice of signs of W_+ and W_- in (4.35).

If $\kappa \ll 1$ and $K > 0$, then

$$\alpha = \operatorname{sgn}(1-\kappa^2)^{1/2} = \operatorname{sgn}\left[1 - \left(\arctan \frac{\bar{\psi}\nu\psi}{\bar{\psi}\psi}\right)^2\right]^{1/2} \tag{4.56}$$

Equation (4.54) corresponds to the positive α, and (4.55) corresponds to the negative α. A transition from one equation to another can take place if

$$\frac{\bar{\psi}\nu\psi}{\bar{\psi}\psi} = \pm\tan 1 \tag{4.57}$$

In this case $|\kappa|=1$ and the last term of equation (4.38) cannot be neglected. Thus, between the regions where ψ satisfies (4.54) and (4.55), there is a region where ψ satisfies (4.38).

In addition, if $|\kappa|=1$ and $|B_{01}|<\infty$ then according to (4.51) j^i becomes null ($c^{-2}j_0^2-j_1^2=0$). In other words, let there be two regions. In one of them equation (4.54) ($\alpha=1$) is fulfilled. In other one equation (4.55) ($\alpha=-1$) is fulfilled. Then by transition from one region into another the vector j^i becomes null, or B_{01} becomes singular. If B_{01} has no singularity anywhere and j^i is timelike everywhere, ψ satisfies one of the equations (4.54), (4.55) everywhere.

Each of equations (4.54) and (4.55) is like a two-dimensional Dirac equation. In fact, the two-dimensional Dirac equation has the form

$$\gamma^j\left(-i\hbar\partial_j-\frac{\varepsilon e}{c}A_j\right)\psi_0+mc\psi_0=0 \tag{4.58}$$

where

$$\gamma^0=\frac{1}{c}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad \gamma^1=\begin{pmatrix}0 & 1\\ -1, & 0\end{pmatrix},\qquad \psi_0=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{4.59}$$

and ψ_1, ψ_2 are complex variables.

Equation (4.58) can be reduced to equation (4.54). For this it is sufficient to substitute the imaginary unit i by τ and to use matrix representation (4.32). Then every complex number z is represented in the form

$$z=\operatorname{Re}z+i\operatorname{Im}z=\operatorname{Re}z+\tau\operatorname{Im}z=\begin{pmatrix}\operatorname{Re}z & \operatorname{Im}z\\ -\operatorname{Im}z & \operatorname{Re}z\end{pmatrix}$$

ψ_0 takes the form

$$\psi_0=\begin{bmatrix}\operatorname{Re}\psi_1 & \operatorname{Im}\psi_1\\ -\operatorname{Im}\psi_1 & \operatorname{Re}\psi_1\\ \operatorname{Re}\psi_2 & \operatorname{Im}\psi_2\\ -\operatorname{Im}\psi_2 & \operatorname{Re}\psi_2\end{bmatrix} \tag{4.60}$$

The matrices $i\gamma^0, i\gamma^1$ are substituted, respectively, by $\tau_1\gamma^0$ and $\tau_1\gamma^1$, where γ^0 and γ^1 have the form (4.36). Each column of the matrix (4.60) is real and satisfies the equation (4.54).

Thus, the continuum of solutions of the two-dimensional Dirac equation (4.57) constitutes a continuum of real solutions of equation (4.54).

Expressions of components of j^i in Dirac's theory coincide with expressions (4.46) of real solutions of equation (4.54). But equation (4.54) has imaginary solutions together with real ones. The imaginary solution can be obtained from the real one by multiplying by the imaginary unit i. Let ψ be a real solution of equation (4.54). It corresponds to a solution ψ_0 of equation (4.58) and to some flux density j^i with $j_0>0$. The imaginary solution $\psi_{\text{im}}=i\psi$ of equation (4.54) corresponds to flux density $(j^i)_{\text{im}}=-j^i$ with $(j_0)_{\text{im}}=-j_0<0$. ψ_{im} does not correspond to any solution of the Dirac equation (4.58). In fact, in Dirac's theory $j_0\geqslant 0$ always, but in the case of equation (4.54) j_0 can be both positive and negative. There is no privileged direction of time. In this sense a set of solutions of equation (4.54) is more abundant than that of the Dirac equation (4.58).

As concerns equation (4.55), it is in some sense equivalent to equation (4.54). In fact, if ψ is a solution of (4.54) then $\psi'=c\gamma^0\gamma^1\psi$ is a solution of (4.55) and vice versa. The same values of j_0, j_1, φ, κ correspond to ψ and ψ', but values of α differ by sign [cf. (4.42)].

Thus, in the case $|\kappa|\ll 1$ [in this case (4.50) is fulfilled] the system (4.4)–(4.8) can be approximated by one of the equations (4.54), (4.55), and the dynamical system state is described by the wave function (4.35) completely. Each of the equations (4.54), (4.55) represents essentially the Dirac equation with the difference that the imaginary unit i is replaced by a real matrix τ_1 that has properties of the imaginary unit i.

In the case when κ is not small but $|\kappa|<1$ (and consequently (4.50) is fulfilled), the system (4.4)–(4.8) can be replaced by equation (4.38) with the wave function ψ describing the dynamical system state completely.

Finally, in the case when $|\kappa|>1$ [and (4.50) is not fulfilled], the system (4.4)–(4.8) can be replaced by equation (4.38), but already the wave function (4.35) does not describe the dynamical system state completely. This fact is connected with that, that according to (4.52) ψ determines the phase κ to within $n\pi$ only. For determination of the n state of the dynamical system it is necessary to give not only ψ, but still, generally speaking, an integer quantity n, which is a certain function of x.

Apparently, n can be determined through ψ given at a certain moment of time, if one uses continuity of κ and $n=0$ inside those regions where the condition (4.50) is fulfilled. But such a determination of n is not local. $n(x)$ is determined by the form of the function ψ, not by the value of ψ at the point x. In other words, n can be a functional of ψ. The sense of ψ is clear. This is a wave function or some analog of it. The sense of the integer variable n is unclear. This problem needs a special investigation. Here I confine myself to the following remark.

In the case when at every point of space-time there are only particles or only antiparticles, the flux density vector j^i is timelike. The condition (4.50) is fulfilled and $n=0$. Violation of (4.50) can arise in that region of space-time where there are both particles and antiparticles. The total flux j^i consists of the particle flux j_p^i and of the antiparticle flux j_a^i:

$$j^i = j_p^i + j_a^i \tag{4.61}$$

Each of the vectors j_p^i and j_a^i is timelike. As far as j_p^0 and j_a^0 have different signs, then in the sum (4.61) the time components are compensated partially or completely. As a result the violation of (4.50) is possible. Then $n \neq 0$ can arise.

It is useful to write the expression (3.19) for the energy density in the case of violation of (4.50). Taking into account (4.17),(4.22),(4.25), and supposing that $K > 0$, one obtains

$$T^0_{\cdot 0} = mc\left(\frac{1-\kappa^2}{(1/c^2)j_0^2 - j_1^2}\right)^{1/2}\left(\frac{1}{c^2}j_0^2 + 2\frac{(1/c^2)j_0^2 - j_1^2}{1-\kappa^2}\kappa^2\right) \tag{4.62}$$

Hence it follows that with fixed j_0, j_1 and $|\kappa| \to \infty$,

$$T^0_{\cdot 0} = mc\left(\frac{(1/c^2)j_0^2}{|(1/c^2)j_0^2 - j_1^2|^{1/2}} + 2\left|\frac{1}{c_2}j_0^2 - j_1^2\right|^{1/2}\right)|\kappa| \tag{4.63}$$

i.e., with large $|\kappa|$ (and consequently large n) the energy density is proportional to n.

It is natural to assume that the energy density is proportional to the number of particles and antiparticles in unit volume (not to j^0, but to $|j_p^0| + |j_a^0|$). Then the fact that $T^0_{\cdot 0} \propto |\kappa|$ with $|\kappa| \to \infty$ can be treated in the sense that the greater $|\kappa|$ is, the more the number of particle and antiparticle taken separately, with their difference fixed. Apparently, κ (and n) represents a quantity of the type $(n_p + n_a)/|n_p - n_a| - 1$. n_p is a concentration of particles, n_a is a concentration of antiparticles. But this problem needs a special investigation.

5. THE TRANSFORMATION PROPERTIES

Let us investigate the law of transformation of ψ with respect to Lorentz transformation

$$t \to t' = t \,\mathrm{Chi}\,\beta + \frac{x}{c}\,\mathrm{shi}\,\beta$$
$$x \to x' = ct\,\mathrm{shi}\,\beta + x\,\mathrm{Chi}\,\beta \tag{5.1}$$

where, Chi and shi denote hyperbolic cosine and hyperbolic sine respectively. β is a parameter of transformation. The following law of transformation of the quantities j^0, j^1, B_{01}, S is obtained:

$$
\begin{aligned}
&j^0 \to j^{0\prime} = j^0 \operatorname{Chi}\beta + \frac{1}{c} j^1 \operatorname{shi}\beta \\
&j^1 \to j^{1\prime} = c j^0 \operatorname{shi}\beta + j^1 \operatorname{Chi}\beta \\
&B_{01} \to B'_{01} = B_{01} \\
&S \to S' = S
\end{aligned}
\tag{5.2}
$$

Using (4.35) and (4.36), the following law of transformation of ψ is obtained:

$$\psi \to \psi' = e^{(1/2)c\gamma^0\gamma^1\beta}\psi \tag{5.3}$$

It is the law by which spinors are transformed (see, for instance, Bogoliubov and Shirkov, 1976). Thus ψ is a spinor.

It is easy to verify that equation (4.38) and each of the equations (4.54), (4.55) are invariant with respect to Lorentz transformation (5.1), (5.3).

Let us consider a transformation of the coordinate reflection

$$t \to t' = -t, \qquad x \to x' = -x \tag{5.4}$$

As far as j^i is a vector and S is a scalar, the following transformation law is obtained:

$$
\begin{aligned}
&j^0 \to j^{0\prime} = -j^0, \qquad j^1 \to j^{1\prime} = -j^1 \\
&B_{01} \to B'_{01} = B_{01}, \qquad S \to S' = S, \qquad K \to K' = K
\end{aligned}
\tag{5.5}
$$

From (4.35), (4.40), (5.5) the following law of transformation of ψ and α follows:

$$\psi \to \psi' = i\psi \tag{5.6}$$

$$\alpha \to \alpha' = -\alpha \tag{5.7}$$

It is easy to verify that equation (4.38) is invariant with respect to transformation (5.4), (5.6), (5.7). As concerns equations (4.54), (4.55), with the transformation (5.4), (5.5) equation (4.54) transforms into (4.55) and (4.55) transforms into (4.54).

Finally, let us consider the transformation that changes the component ε of the world line orientation. It is connected with the change of the sign of a parameter that numbers points of a world line. This transformation does not affect a coordinate system and can be represented in the form

$$\varepsilon \rightarrow \varepsilon' = -\varepsilon, \qquad \psi \rightarrow \psi' = c\gamma^1\gamma^0\tau_0\psi$$
$$x^i \rightarrow x'^i = x^i \tag{5.8}$$

This transformation leads to

$$j^0 \rightarrow j^{0\prime} = j^0, \qquad j^1 \rightarrow j^{1\prime} = j^1 \tag{5.9}$$

$$\tan(2\varphi) \rightarrow \tan(2\varphi') = -\tan(2\varphi), \qquad \tan\kappa \rightarrow \tan\kappa' = -\tan\kappa \tag{5.10}$$

Let us suppose that (5.10) can be replaced by

$$\varphi \rightarrow \varphi' = -\varphi, \qquad \kappa \rightarrow \kappa' = -\kappa \tag{5.11}$$

Then it follows from (4.6), (4.17) that

$$p_0 \rightarrow p_0' = -p_0, \qquad p_1 \rightarrow p_1' = -p_1$$

Applying (5.8), (5.10) to (4.35), one obtains

$$W_+ \rightarrow W_+' = -W_+, \qquad W_- \rightarrow W_-' = W_- \tag{5.12}$$

From (4.40)

$$\alpha \rightarrow \alpha' = -\alpha \tag{5.13}$$

It follows from (5.8), (5.11), (5.13) that equation (4.38) is invariant with respect to transformation (5.8). With transformation (5.8) equation (4.54) transforms into (4.55) and vice versa. The transformation (5.8) changes the sign of the electric charge εe of the particle, i.e., a particle is transformed into antiparticle and vice versa. For this reason the transformation (5.8) is associated with a charge conjugation transformation.

6. DISCUSSION

Investigation of the solution of equation (4.38) is a complicated and difficult problem. I hope soon to make some attempts to investigate equation (4.38) more. Here I confine myself to the following remark.

If the electromagnetic field is absent, the steady-state solution of the (4.38) can be represented in the form

$$\psi=\exp\left[(\tau_1/\hbar)(p_0x^0+p_1x^1)\right](\gamma^i p_i-\alpha mc)W_0 \tag{6.1}$$

where p_0, p_1 are real constants satisfying the relation

$$\frac{1}{c^2}p_0^2-p_1^2=m^2c^2 \tag{6.2}$$

and W_0 is a column of four arbitrary numbers that are real or imaginary simultaneously. Expression (6.1) is a solution of the two-dimensional Dirac equation (4.53). An arbitrary linear combination of solutions (6.1) is a solution of (4.53) but, generally speaking, it is not a solution of the (4.38). In other words, for nonlinearity of the equation (4.38), its steady-state solutions (6.1) interact. Formally it manifests itself in the fact that the term $\alpha Mc\psi$ of equation (4.38) vanishes for the solution (6.1), but, generally speaking, it is not equal to zero for a linear combination of expressions (6.1). This term describes an interaction of the plane waves (6.1). The interaction is connected with the presence of a gradient of the flux density j^i (κ becomes different from zero). It arises whenever j^i depends on a coordinate or on time.

The main result of the paper is the following proposition. In the two-dimensional space-time a system of equations for tensors of integer rank can be written that can be approximated with a linear spinor equation (Dirac equation). This circumstance is surprising. The spinor fields are supposed to differ from the tensor fields strongly. For instance, particles described by a spinor field obey Fermi–Dirac statistics, whereas particles described by a tensor one obey Bose–Einstein statistics. The spinor Dirac equation is considered usually as an elementary equation that cannot be reduced to anything simpler. Here one discovers that an equation that differs from the Dirac equation very slightly can be obtained starting from the relativistic statistics of world lines.

Strictly speaking, the possibility of approximation of the system (3.9)–(3.11) by means of a linear spinor equation is shown for the two-dimensional space-time only. The possibility of such an approximation in the case of four-dimensional space-time remains open. The question remains open, if the system (3.9)–(3.12) describes an ensemble of particles with spin $\frac{1}{2}$.

Such a possibility seems probable, if one takes into account two circumstances. (1) In the nonrelativistic approximation and in the case where p_i has a potential, the system (3.9)–(3.12) is equivalent to the Schrödinger equation. (2) In the two-dimensional space-time the system (3.9)–(3.12) can be approximated by the Dirac equation.

APPENDIX

Let us show that the equation

$$\Phi^j\left(g_{jk}-\frac{\lambda^2}{4K^2}g_{il}B^i_{\cdot j}B^l_{\cdot k}\right)\Phi^k=0 \tag{A.1}$$

for vector Φ^j has real solutions. Here

$$K^2=j^ij_i+\frac{\lambda^2}{8}B^i_{\cdot l}B^l_{\cdot i} \tag{A.2}$$

For the proof of this fact it is convenient to decompose vector Φ^j over eigenvectors of the matrix $B^i_{\cdot k}$ in the form

$$\Phi^j=\sum_{l=1}^{4}a_{(l)}u^j_{(l)} \tag{A.3}$$

where $u^j_{(l)}$ are eigenvectors of matrix $B^j_{\cdot k}$

$$B^j_{\cdot k}u^k_{(l)}=\lambda_{(l)}u^j_{(l)} \tag{A.4}$$

$\lambda_{(l)}$ are eigenvalues. There are no summations over indices inside parentheses. The matrix $B^j_{\cdot k}$ can be represented in the form

$$B^j_{\cdot k}=\left\|\begin{matrix} 0 & E_1/c & E_2/c & E_3/c \\ cE_1 & 0 & H_3 & -H_2 \\ cE_2 & -H_3 & 0 & H_1 \\ cE_3 & H_2 & -H_1 & 0 \end{matrix}\right\| \tag{A.5}$$

where $\mathbf{E}=\{E_1, E_2, E_3\}$, $\mathbf{H}=\{H_1, H_2, H_3\}$ are certain vectors. A calculation gives the following for the eigenvalues $\lambda_{(l)}$ and eigenvectors $u^j_{(l)}$:

$$\lambda_{(1)}=-\lambda_{(2)}=\left\{\frac{\left[(\mathbf{E}^2-\mathbf{H}^2)^2+4(\mathbf{EH})^2\right]^{1/2}+\mathbf{E}^2-\mathbf{H}^2}{2}\right\}^{1/2}$$

$$\lambda_{(3)}=-\lambda_{(4)}=i\nu=i\left\{\frac{\left[(\mathbf{E}^2-\mathbf{H}^2)^2+4(\mathbf{EH})^2\right]^{1/2}-(\mathbf{E}^2-\mathbf{H}^2)}{2}\right\}^{1/2} \tag{A.6}$$

$$u^j_{(1)}=\left\{\frac{1}{c},\frac{1}{\sigma_1}(\lambda_{(1)}\mathbf{E}+\nu'\mathbf{H}+[\mathbf{E}\times\mathbf{H}])\right\}$$

$$u^j_{(2)}=\left\{\frac{1}{c},\frac{1}{\sigma_1}(-\lambda_{(1)}\mathbf{E}-\nu'\mathbf{H}+[\mathbf{E}\times\mathbf{H}])\right\}$$

$$u^j_{(3)}=\left\{\frac{1}{c},\frac{1}{\sigma_3}(i\nu\mathbf{E}-i\lambda'_1\mathbf{H}+[\mathbf{E}\times\mathbf{H}])\right\}$$

$$u^j_{(4)}=\left\{\frac{1}{c},\frac{1}{\sigma_3}(-i\nu\mathbf{E}+i\lambda'_1\mathbf{H}+[\mathbf{E}\times\mathbf{H}])\right\} \tag{A.7}$$

$$\lambda'_1=\lambda_{(1)}\operatorname{sgn}(\mathbf{EH}),\qquad \nu'=\nu\operatorname{sgn}(\mathbf{EH}) \tag{A.8}$$

$$\sigma_1=\frac{\mathbf{E}^2+\mathbf{H}^2+\left[(\mathbf{E}^2-\mathbf{H}^2)^2+4(\mathbf{EH})^2\right]^{1/2}}{2}$$

$$\sigma_3=\frac{\mathbf{E}^2+\mathbf{H}^2-\left[(\mathbf{E}^2-\mathbf{H}^2)^2+4(\mathbf{EH})^2\right]^{1/2}}{2} \tag{A.9}$$

where $(\mathbf{EH})$ and $[\mathbf{E}^2\times\mathbf{H}]$ are, respectively, a scalar product and a vector one. Eigenvectors $u^j_{(1)}, u^j_{(2)}$ are real, but $u^j_{(3)}$ and $u^j_{(4)}$ are complex with

$$\left(u^j_{(3)}\right)^*=u^j_{(4)} \tag{A.10}$$

Here (*) denotes a complex conjugation. If Φ^j is real, then $a_{(1)}, a_{(2)}$ are real, but $a_{(3)}, a_{(4)}$ are complex with

$$a^*_{(3)}=a_{(4)} \tag{A.11}$$

Substituting (A.3) into (A.1) one obtains

$$\sum_{l,l'=1}^{4} a_{(l)}a_{(l')}\left(1-\frac{\lambda^2}{4K^2}\lambda_{(l)}\lambda_{(l')}\right)\left(u^j_{(l)}g_{jk}u^k_{(l')}\right)=0 \tag{A.12}$$

Taking into account (A.11), and the fact that

$$u^j_{(l)}g_{jk}u^k_{(l')}=\begin{cases}2\left(1-\dfrac{\sigma_3}{\sigma_1}\right) & \text{if } l=1, l'=2 \text{ or } l=2, l'=1\\[2ex] 2\left(1-\dfrac{\sigma_1}{\sigma_3}\right) & \text{if } l=3, l'=4 \text{ or } l=4, l'=3\\[2ex] 0 & \text{in other cases}\end{cases} \tag{A.13}$$

and that by means of (A.6) K^2 can be represented in the form

$$K^2 = j^i j_i + \frac{\lambda^2}{4}\left(\lambda_{(1)}^2 - |\lambda_{(3)}|^2\right) \tag{A.14}$$

one obtains from (A.12)

$$a_{(1)} a_{(2)} = |a_{(3)}|^2 \frac{\sigma_1}{\sigma_3} \frac{j^i j_i + \lambda\lambda_{(1)}^2/4}{j^i j_i - \lambda\lambda_{(3)}^2/4} \tag{A.15}$$

It follows from (A.15) that for an arbitrary real $a_{(2)} \neq 0$ the $a_{(1)}$ is real. Thus, (A.1) has real solutions depending on three real parameters: $a_{(2)}, \operatorname{Re} a_{(3)}, \operatorname{Im} a_{(3)}$.

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